Diffusion phenomena for partially dissipative hyperbolic systems

Jens Wirth

Abstract.

In this note we provide some precise estimates explaining the diffusive structure of partially dissipative systems with time-dependent coefficients satisfying a uniform Kalman rank condition. Precisely, we show that under certain (natural) conditions solutions to a partially dissipative hyperbolic system are asymptotically equivalent to solutions of a corresponding parabolic equation.

The approach is based on an elliptic WKB analysis for small frequencies in combination with exponential stability for large frequencies due to results of Beauchard and Zuazua and arguments of perturbation theory.

§1. Introduction

The classical diffusion phenomenon observed by Hsiao and Liu [2] and later Nishihara [6] provides an asymptotic equivalence for solutions to damped wave equations (or porous media equations) and corresponding solutions to the heat equation. To be precise, for any solution to the Cauchy problem for the damped wave equation

$$u_{tt} - \Delta u + u_t = 0, \qquad u(0,\cdot) = u_1, \quad u_t(0,\cdot) = u_2 \tag{1}$$

on $\mathbb{R}^n$ we find a corresponding solution to the heat equation

$$w_t = \Delta w, \qquad w(0,\cdot) = w_0 = u_1 + u_2, \tag{2}$$

such that their difference satisfies

$$\|u(t,\cdot) - w(t,\cdot)\|_2 \le Ct^{-1}\big(\|u_1\|_2 + \|u_2\|_2\big). \tag{3}$$

On the other hand, both solutions do not decay in general. By this we mean, that the estimates

$$\|u(t,\cdot)\|_2 \le C\big(\|u_1\|_2 + \|u_2\|_2\big), \qquad \|w(t,\cdot)\|_2 \le C\|w_0\|_2 \tag{4}$$

are both sharp. The diffusion phenomenon makes the diffusive nature of dissipation apparent. See also [4] and [7], [5] for a detailed study of estimates in this spirit.

Our aim is to show that this statement has a natural counterpart in the language of differential hyperbolic systems with time-dependent coefficients. Our main tools will be a block-diagonalisation of the (full) symbol for small frequencies combined with an exponential decay rate for large frequencies deduced from methods of [8] and [1].

Key words and phrases. partially dissipative systems, a priori estimates, diffusion phenomenon.

§2. Main results

Before stating the main results, we will introduce the classes of coefficient functions we are going to use later on. For a parameter $r \in \mathbb{R}$ we denote by $\mathcal{T}\{r\}$ the class of C^∞-functions on $a : \mathbb{R}_+ \to \mathbb{C}$ satisfying the symbolic estimates

$$\left|\partial_t^k a(t)\right| \le C_k (1+t)^{-r-k}. \tag{5}$$

We consider a Cauchy problem of the form

$$\mathrm{D}_t U = \sum\nolimits_{k=1}^n A_k(t)\mathrm{D}_{x_k} U + \mathrm{i}B(t)U, \qquad U(0,\cdot) = U_0 \tag{6}$$

for an unknown vector-valued function $U : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}^d$ and with coefficient matrices

$$A_k(t), B(t) \in \mathcal{T}\{0\} \otimes \mathbb{C}^{d\times d} \tag{7}$$

satisfying the following assumptions:

(B1): the matrices $A_k(t)$ are self-adjoint;

(B2): the matrix $B(t)$ is non-negative and $0 \in \operatorname{spec} B(t)$ is a simple eigenvalue uniformly separated from the remaining part of the spectrum,

$$\operatorname{dist}\big(0, \operatorname{spec} B(t) \setminus \{0\}\big) \ge \gamma > 0; \tag{8}$$

and

(B3): the matrices $B(t)$ and $A(t,\xi) = \sum_{k=1}^n A_k(t)\xi_k$ satisfy

$$\frac{1}{c}\|v\|^2 \le \sum\nolimits_{j=0}^{d-1} a_k \|B(t)A^j(t,\xi)v\|^2 \le c\|v\|^2, \qquad t \ge t_0 \tag{9}$$

for any choice of numbers $a_0, \ldots, a_{d-1} > 0$ and suitable constants c and t_0 depending on them.

Assumption (B1) guarantees that the system (6) is hyperbolic. The non-negativity of $B(t)$ implies dissipativity. The non-triviality of the null space of $B(t)$ in (B2) is not the crucial assumption as this can always be achieved by multiplying $U(t,x)$ with a scalar function and the main result remains true modulo the obvious modifications. Crucial point for our result is that the lowest eigenvalue is uniformly separated from the others.

The last assumption is used to treat non-zero frequencies. We refer to (B3) as uniform Kalman rank condition. It allows to construct a Lyapunov functional in order to prove that the Fourier transform of solutions to (6) with respect to spacial variables satisfies

$$\|\widehat{U}(t,\cdot)\|_2 \le C\mathrm{e}^{-ct[\xi]^2}\|\widehat{U}_0\|_2, \qquad [\xi] = |\xi|/\langle\xi\rangle \simeq \min\{|\xi|, 1\}, \tag{10}$$

with suitable non-negative constants $C, c > 0$. The proof for this fact is essentially the same one given in [1] with the obvious modification that the Lyapunov functional has coefficients in $\mathcal{T}\{0\}$ such that heir argument works only if t is large.

If assumptions (B1)–(B3) are satisfied then we can associate to (6) a parabolic equation

$$\partial_t w = \nabla \cdot \alpha(t)\nabla w + \beta(t)\cdot\nabla w + \gamma(t) w, \qquad w(t_0,\cdot) = w_0 \tag{11}$$

with coefficients $\alpha(t) \in \mathcal{T}\{0\} \otimes \mathbb{C}^{d\times d}$, $\beta(t) \in \mathcal{T}\{1\} \otimes \mathbb{C}^d$ and $\gamma(t) \in \mathcal{T}\{1\}$, a smoothing linear operator mapping $W : U_0 \mapsto w_0$ and a vector $K(t,\mathrm{D})$ of second order differential operators with coefficients in $\mathcal{T}\{0\}$ such that the following theorem holds true.

Theorem 1. *For any solution $U(t,x)$ to (6) to initial data $U_0 \in L^2(\mathbb{R}^n;\mathbb{C}^d)$ the corresponding solution $w(t,x)$ to (11) with $w_0 = WU_0$ satisfies*

$$\|U(t,\cdot) - K(t,\mathrm{D})w(t,\cdot)\|_2 \le Ct^{-1/2}\log(t)\|U_0\|_2, \qquad t \ge t_0. \tag{12}$$

The coefficients of the parabolic problem (11) as well as of the operators W and $K(t,\mathrm{D})$ will be made precise in the following section. They arise naturally within a decoupling procedure of the system.

§3. Proof

The proof is divided into several parts. First we will asymptotically block-diagonalise the full symbol of the problem (6) for small frequencies within the so-called elliptic zone $\mathcal{Z}_{\mathrm{ell}}(\epsilon) = \{(t,\xi) : t \ge \epsilon^{-1}, |\xi| \le \epsilon\}$ for sufficiently small ϵ. Similar to [10], this will allow us to obtain asymptotic formulas for the fundamental solution for bounded $t|\xi| \le \delta$ within this zone. In a second step we will use main terms of this asymptotic representation to deduce the coefficients of (11). In a final step we will prove the desired estimate of Theorem 1 chosing k and δ sufficiently large. For full details and further information we refer to [9, Chapter 5].

3.1. Block-diagonalising $A(t,\xi) + B(t)$

The consideration follows partly [3] and [11]. In the sequel we will use the notation

$$\mathcal{P}\{m\} = \left\{ p(t,\xi) = \sum\nolimits_{|\alpha|\le m} p_\alpha(t)\xi^\alpha : p_\alpha(t) \in \mathcal{T}\{m-|\alpha|\} \right\} \tag{13}$$

for polynomials of degree m with coefficients in the $\mathcal{T}$-classes. We will use the same notation for the matrix-valued case. Clearly $\mathcal{P}\{m+1\} \subset \mathcal{P}\{m\}$ such that these classes constitute a hierarchy. Condition (7) guarantees that $A(t,\xi) \in \mathcal{P}\{1\}$, while $B(t) \in \mathcal{P}\{0\}$. Furthermore, due to (8) we also know that there exists an invertible matrix $M(t) \in \mathcal{P}\{0\}$ such that

$$M^{-1}(t)B(t)M(t) = \text{b-diag}_{(1,d-1)}\left(0, \tilde{B}(t)\right) = \mathcal{D}(t) \tag{14}$$

is block-diagonal. Denoting $V^{(0)}(t,\xi) = M^{-1}(t)\widehat{U}(t,\xi)$, $\widehat{U}$ being the partial Fourier transform of U with respect to spatial variables, we obtain for $V^{(0)}$ the ordinary differential equation

$$\mathrm{D}_t V^{(0)} = \big(\mathcal{D}(t) + R_1(t,\xi)\big)V^{(0)} \tag{15}$$

with $R_1(t,\xi) = M^{-1}(t)A(t,\xi)M(t) + \big(\mathrm{D}_t M^{-1}(t)\big)M(t) \in \mathcal{P}\{1\}$ of lower order in the $\mathcal{P}$-hierarchy. By induction we show the following lemma.

Lemma 2. *Let $k \in \mathbb{N}$, $k \ge 1$. Then there exists a constant c_k and matrices $N_k(t,\xi) \in \mathcal{P}\{0\}$, block-diagonal $F_k(t,\xi) \in \mathcal{P}\{1\}$ and $R_{k+1}(t,\xi) \in \mathcal{P}\{k+1\}$ such that*

$$\big(\mathrm{D}_t - \mathcal{D}(t) - R_1(t,\xi)\big)N_k(t,\xi) = N_k(t,\xi)\big(\mathrm{D}_t - \mathcal{D}(t) - F_k(t,\xi) - R_{k+1}(t,\xi)\big) \tag{16}$$

holds true within $\mathcal{Z}_{\mathrm{ell}}(c_k)$. The matrix $N_k(t,\xi)$ is uniformly invertible within this zone.

Proof. We explain the $k=1$ step in detail. We construct a matrix $N^{(1)}(t,\xi)\in\mathcal{P}\{1\}$ such that

$$\begin{aligned}(17)\quad &\big(\mathrm{D}_t-\mathcal{D}(t)-R_1(t,\xi)\big)(\mathrm{I}+N^{(1)}(t,\xi))\\ &\qquad -(\mathrm{I}+N^{(1)}(t,\xi))\big(\mathrm{D}_t-\mathcal{D}(t,\xi)-F_1(t,\xi)\big)\in\mathcal{P}\{2\}\end{aligned}$$

holds true for some diagonal matrix $F_1(t,\xi)\in\mathcal{P}\{1\}$. Collecting all terms not belonging to the right class yields again conditions for the matrices $N^{(1)}(t,\xi)$ and $F_1(t,\xi)$. Indeed,

$$[\mathcal{D}(t),N^{(1)}(t,\xi)]=-R_1(t,\xi)+F_1(t,\xi) \tag{18}$$

must be satisfied and therefore it is reasonable to set

$$F_1(t,\xi)=\text{b-diag}_{(1,d-1)}\,R_1(t,\xi) \tag{19}$$

and to denote by $N^{(1)}(t,\xi)=\begin{pmatrix}0 & n_{1,1}(t,\xi)^\top\\ n_{1,2}(t,\xi) & 0\end{pmatrix}$ the matrix with entries

$$n_{1,j}(t,\xi)=\int_0^\infty \mathrm{e}^{-s\tilde{B}(t)}\tilde{R}_{1,j}(t,\xi)\mathrm{d}s,\qquad R_1-F_1=\begin{pmatrix}0 & \tilde{R}_{1,1}^\top\\ \tilde{R}_{1,2} & 0\end{pmatrix}, \tag{20}$$

solving Sylvester's equation (18). This implies that $N^{(1)}(t,\xi),F_1(t,\xi)\in\mathcal{P}\{1\}$. Recursively, we will then construct matrices $N^{(k)}(t,\xi)\in\mathcal{P}\{k\}$ and $F^{(k)}(t,\xi)\in\mathcal{P}\{k\}$ block-diagonal, such that for

$$N_K(t,\xi)=\mathrm{I}+\sum\nolimits_{k=1}^K N^{(k)}(t,\xi),\qquad F_K(t,\xi)=\sum\nolimits_{k=1}^K F^{(k)}(t,\xi), \tag{21}$$

the estimate

$$\begin{aligned}(22)\quad B_K(t,\xi)=&\big(\mathrm{D}_t-\mathcal{D}(t)-R_1(t,\xi)\big)N_K(t,\xi)\\ &\qquad -N_K(t,\xi)\big(\mathrm{D}_t-\mathcal{D}(t)-F_K(t,\xi)\big)\in\mathcal{P}\{K+1\}\end{aligned}$$

is valid. As we just did this for $K=1$, it remains to do the recursion $k\mapsto k+1$. Assume $B_k(t,\xi)\in\mathcal{P}\{k+1\}$. The requirement to be met is that

$$B_{k+1}(t,\xi)-B_k(t,\xi)=-[\mathcal{D}(t),N^{(k+1)}(t,\xi)]+F^{(k+1)}(t,\xi)\mod\mathcal{P}\{k+2\}, \tag{23}$$

which yields again a Sylvester equation and its solution is given in analogy to the above case. It is evident that the construction implies $F^{(k+1)}(t,\xi),N^{(k+1)}(t,\xi)\in\mathcal{P}\{k+1\}$ together with $B_{k+1}(t,\xi)\in\mathcal{P}\{k+2\}$. The matrices $N_k(t,\xi)\in\mathcal{P}\{0\}$ are invertible with inverse $N_k^{-1}(t,\xi)\in\mathcal{P}\{0\}$ if we restrict our consideration to a sufficiently small elliptic zone $\mathcal{Z}_{\mathrm{ell}}(c_k)$. Q.E.D.

3.2. Asymptotic integration

We consider the transformed problem in $V^{(k)}(t,\xi)=N_k(t,\xi)V^{(0)}(t,\xi)$,

$$\mathrm{D}_tV^{(k)}(t,\xi)=\big(\mathcal{D}(t)+F_k(t,\xi)+R_{k+1}(t,\xi)\big)V^{(k)}(t,\xi), \tag{24}$$

and reformulate this as integral equation for its fundamental solution. Denoting it by $\mathcal{E}_k(t,s,\xi)$, we know that it solves the above equation to initial data $\mathcal{E}_k(s,s,\xi)=\mathrm{I}\in\mathbb{C}^{d\times d}$.

Let $\Theta_k(t,s,\xi)$ be the fundamental solution to the block-diagonalised system $\mathrm{D}_t - \mathcal{D}(t) - F_k(t,\xi)$. Then

$$\Theta_k(t,s,\xi) = \begin{pmatrix} \Xi_k(t,s,\xi) & 0 \\ 0 & \tilde{\Theta}_k(t,s,\xi) \end{pmatrix}, \qquad \|\tilde{\Theta}_k(t,s,\xi)\| \lesssim \mathrm{e}^{-\gamma\frac{t-s}{2}} \tag{25}$$

holds true for $t \geq s$ uniformly within $\mathcal{Z}_{\mathrm{ell}}(\epsilon)$ for $\epsilon \leq c_k$ sufficiently small. Furthermore, $\mathcal{E}_k(t,s,\xi)$ satisfies

$$\mathcal{E}_k(t,s,\xi) = \Theta_k(t,s,\xi) + \int_s^t \Theta_k(t,\theta,\xi) R_{k+1}(\theta,\xi)\mathcal{E}_k(\theta,s,\xi)\mathrm{d}\theta. \tag{26}$$

We solve this Volterra integral equation using the Neumann series

$$\begin{aligned}\mathcal{E}_k(t,s,\xi) = \Theta_k(t,s,\xi) + \sum_{\ell=1}^{\infty} \int_s^t \Theta_k(t,t_1,\xi) R_{k+1}(t_1,\xi) \int_s^{t_1} \cdots \\ \cdots \int_s^{t_{\ell-1}} \Theta_k(t_{\ell-1},t_\ell,\xi) R_{k+1}(t_\ell,\xi)\mathrm{d}t_\ell \cdots \mathrm{d}t_1.\end{aligned} \tag{27}$$

This series converges based on the estimates for the remainder term $R_{k+1}(t,\xi) \in \mathcal{P}\{k+1\}$. It can be estimated by

$$\|\mathcal{E}_k(t,s,\xi)\| \leq \exp\left(\int_s^t \|R_{k+1}(\theta,\xi)\|\mathrm{d}\theta\right). \tag{28}$$

The series converges uniformly within $\mathcal{Z}_{\mathrm{ell}}(c_k) \cap \{t|\xi|^{(k+1)/2} \leq \delta\}$ for any constant δ and $\|\mathcal{E}_k(t,s,\xi) - \Theta_k(t,s,\xi)\| \to 0$ as $c_k \to 0$ for fixed $\delta > 0$ as soon as we choose $k \geq 1$.

We can do slightly better. Let $W(s) = \lim_{t\to\infty} e_1^\top \mathcal{E}_k(t,s,0)$. Then the following lemma holds true. The full proof can be found in [9].

Lemma 3. *The fundamental solution $\mathcal{E}_k(t,s,\xi)$, k sufficiently large, satisfies the estimate*

$$\|\mathcal{E}_k(t,s,\xi) - \Xi_k(t,s,\xi)e_1 W_2(s)\| \leq C_k(1+t)^{-\frac{1}{2}} \tag{29}$$

uniformly in $|\xi| \leq \epsilon_k$.

3.3. The parabolic reference problem

The parabolic reference problem is defined in terms of the upper left corner entry of the matrix $F_k(t,\xi)$ modulo terms from $\mathcal{P}\{3\}$,

$$f_1^{(k)}(t,\xi) = \mathrm{i}\sum_{i,j=1}^{d} \alpha_{i,j}(t)\xi_i\xi_j + \sum_{i=1}^{d} \beta_i(t)\xi_i + \gamma(t) \mod \mathcal{P}\{3\} \tag{30}$$

with $\alpha_{i,j}(t) \in \mathcal{T}\{0\}$, $\beta_i(t) \in \mathcal{T}\{0\}$ and $\gamma(t) \in \mathcal{T}\{1\}$. Now the uniform Kalman rank condition (B3) implies a structural property. It follows that $\beta_i(t)$ has to decay, $\beta_i(t) \in \mathcal{T}\{1\}$, and also that the quadratic matrix $(\alpha_{i,j}(t))_{i,j}$ is positive definite modulo $\mathcal{T}\{1\}$. The latter is a direct consequence of the estimate (10).

This defines the coefficients $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ in (11). The corner entry $\Xi_2(t,s,\xi)$ of $\Theta_2(t,s,\xi)$ is the fundamental solution to this Cauchy problem.Furthermore, we define $K(t,\xi) = M(t)N_2(t,\xi)e_1$, where $e_1 = (1,0,\ldots,0)^\top \in \mathbb{C}^d$. By definition we have $K(t,\xi) \in \mathcal{P}\{2\}$.

3.4. The final estimate

We define $w_0 = WU_0$ in such a way that we cancel the main term of the solution within $\mathcal{Z}_{\mathrm{ell}}(c_k) \cap \{t|\xi| \le \delta\}$, i.e., we define

$$\widehat{w}_0 = W_2(t_0,\xi) N_2^{-1}(t_0,\xi) M^{-1}(t_0) \mathcal{E}(t_0,0,\xi)\chi(\xi)\widehat{U}_0 \tag{31}$$

with $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$, $\chi(\xi) = 1$ near $\xi = 0$ and $\operatorname{supp}\chi \subset B_{c_2}(0)$. Then the estimates of Lemma 3 imply the following statement. The logarithmic term is caused by comparing $\Xi_k(t,s,\xi)$ with $\Xi_2(t,s,\xi)$.

Lemma 4. *The solution $w(t,x)$ to* (11) *satisfies*

$$\|\widehat{U}(t,\cdot) - K(t,\cdot)\widehat{w}(t,\cdot)\|_2 \le C'(1+t)^{-1/2}\log(\mathrm{e}+t). \tag{32}$$

§4. Concluding remarks

We used that $\ker B(t)$ is one-dimensional. The result can be generalised to situations where a (stable) higher-dimensional null-space appears and *further* diagonalisability conditions on appearing lower order terms in the diagonalisation hierarchy are satisfied. This corresponds to the block-diagonalisation in [3] and again yields a parabolic reference system for these diffusive modes.

If assumption (B3) is violated, a variety of other asymptotic scenarios may appear. If $n = 1$, the results of [1] hint to a decomposition of solutions into traveling waves, parabolic type modes and exponentially decaying modes.

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J. Wirth, IADM, FB Mathematik, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: jens.wirth@iadm.uni-stuttgart.de